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CONVERGENCE OF APPROXIMATE SOLUTIONS FOR WAVES IN A STRATIFIED FLUID WITH APPROXIMATED DENSITY DISTRIBUTION*

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Equations defining in linear approximation the wave motions in an arbitrarily stratified fluid are derived. Investigation of convergence of wave solutions with approximations $\rho_n(z)$ of the mean density profile $\rho_0(z)$ shows that uniform convergence of $\rho_n(z)$ to $\rho_0(z)$ is the sufficient condition of convergence of wave equation solutions. The convergence of solutions is uniform on sets of upper bound wave numbers and lower bound phase velocities of waves. Examples that show that when the continuous function $\rho_0(z)$ is approximated by step-wide functions $\rho_n(z)$ the convergence of solutions for internal waves is not uniform over the whole set of admissible wave numbers and phase velocities of waves.

1. Unsteady wave motions of a horizontally unbounded layer of perfect incompressible fluid whose density in the steady state depends only on the vertical coordinate z is defined in linear approximation by the boundary value problem /1/, from which in /1,2/ were derived for the vertical velocity component w the following equation and boundary conditions

$$\frac{\partial^3}{\partial t^2\,\partial z}\left(\rho_0\frac{\partial w}{\partial z}\right)+\left(\rho_0\frac{\partial^2}{\partial t^2}-g\frac{d}{dz}\rho_0\right)\Delta_2 w=0 \tag{1.1}$$

$$\left(\frac{\partial^3}{\partial^2 t\,\partial z}-g\Delta_2\right)w=0\ (z=0),\quad w=0\quad (z=-H);\quad \Delta_2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

where $\rho_0=\rho_0(z)$ is the fluid density at equilibrium, the z axis is directed vertically upward, H is the fluid depth, and g is the free fall acceleration.

Elementary modes of motion in the case of small perturbations can be determined by setting

$$w=W(z)\exp\left[i\left(\mathbf{r}\cdot\mathbf{x}-\sigma t\right)\right];\ \mathbf{x}=(x,y) \tag{1.2}$$

where $\mathbf{r}=(\mu,\nu)$ is the horizontal wave vector and σ is the frequency. The substitution of this expression into (1.1) yields the boundary value problem in eigenvalues with parameter $r^2=\mu^2+\nu^2$

$$\frac{d}{dz}\left(\rho_0\frac{dW}{dz}\right)-\left(\rho_0+g\sigma^{-2}\frac{d}{dz}\rho_0\right)r^2W=0 \tag{1.3}$$

$$dW/dz-g\sigma^{-2}r^2W=0\ (z=0),\quad W=0\ (z=-H)$$

whose solution enables us to establish the dispersion formulas $c=c(r)$, $c=\sigma r^{-1}$ for the phase velocity of wave propagation (1.2).

We assume in what follows that the fluid is stably stratified, i.e. $\rho_0(z)$ is a monotonically nonincreasing function and $\rho_0(z)>0$. When $\rho_0(z)$ is not everywhere differentiable, it has to be assumed that Eq.(1.3) is satisfied for all $z\in(-H,0)$ for which $\rho_0(z)$ has a derivative, and the boundary conditions are to be supplemented by conditions of continuity of function $W(z)$ and of total pressure /1/

$$W(z),\quad \rho_0(z)\left[\frac{d}{dz}-gc^{-2}\right]W\in C(-H,0) \tag{1.4}$$

Substitution of the independent variables

$$s=\int_{-H}^{z}\rho_0^{-1}(\xi)\,d\xi,\qquad ds=\frac{dz}{\rho_0(z)},\qquad z=z(s)$$

reduces problem (1.3) with condition (1.4) to the form

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$$\frac{d^2W_1}{ds^2} - \left[\rho_1^2 + gc^{-2}\frac{d\rho_1}{ds}\right] r^2 W_1 = 0 \tag{1.5}$$

$$dW_1/ds - gc^{-2}\rho_1 W_1 = 0 \quad (s = b), \quad W_1 = 0 \quad (s = 0) \tag{1.6}$$

$$W_1, \quad dW_1/ds - gc^{-2}\rho_1 W \in C\,(0,\, b) \tag{1.7}$$

$$\left(W_1(s) = W(z(s)), \qquad \rho_1(s) = \rho_0(z(s)), \qquad b = \int_{-H}^{0} \rho_0^{-1}(\xi)\, d\xi\right)$$

Integrating (1.5) twice from 0 to s and taking into account the boundary condition at the bottom, we obtain an equation of the Volterra type. The first integration yields

$$\frac{dW_1(s)}{ds} = \frac{dW_1(0)}{ds} + \int_0^s W_1(\alpha)\,[r^2\rho_1^2(\alpha)\,d\alpha + gc^{-2}\,d\rho_1(\alpha)] \tag{1.8}$$

and after second integration we have

$$W_1(s) = s\frac{dW_1(0)}{ds} + \int_0^s W_1(\alpha)(s-\alpha)\,[r^2\rho_1^2(\alpha)\,d\alpha + gc^{-2}\,d\rho_1(\alpha)] \tag{1.9}$$

Equations (1.8) and (1.9) retain their meaning also then when function $\rho_1(s)$ is not everywhere differentiable and has discontinuities. It can be shown that when function $W_1(s)$ is continuous, the second of conditions (1.7) follows from formulas (1.8) and (1.9). Thus Eqs. (1.8) and (1.9) are to be considered only with the first of conditions (1.6).

2. For the equation

$$W(s) = \varphi(s) + \int_0^s (s-\alpha)\, W(\alpha)\, d\Phi(\alpha) \qquad (0 \leqslant s \leqslant b) \tag{2.1}$$

more general than (1.9) we have the following theorem.

Theorem 1. Let $\varphi(s)$ be a continuous and $\Phi(s)$ a continuous from the right function with limited variation on the segment $[0, b]$. Then Eq.(2.1) has a unique solution in the class of continuous functions on segment $[0, b]$.

Proof. Proof of such theorem is given in /3/ (see Theorem 11.2.1) for the case when $\varphi(s)$ is a linear function. Since uniqueness of solution of Eq.(2.1) in the case of an arbitrary continuous function $\varphi(s)$ does not differ from that considered in /3/, we shall prove only the existence of solution.

We assume, in conformity with /3/, that in Eq.(2.1) $\Phi(s)$ is a step function with a finite number of discontinuities at points s_k, where $0 < s_1 < \ldots < s_n < b$. For such function $\Phi(s)$ the solution of Eq.(2.1) is of the form

$$W(s) = \varphi(s) + \sum_{s_i < s} W(s_i)(s - s_i)\,[\Phi(s_i) - \Phi(s_i - 0)] \tag{2.2}$$

Taking in equality (2.2) absolute values, we obtain the estimates

$$\begin{gathered} |\,W(s)\,| \leqslant F_{k+1} \text{ for } s_k \leqslant s \leqslant s_{k+1} \; (k = 0, 1, \ldots, n;\; s_0 = 0, \\ s_{n+1} = b) \\ F_{k+1} = N + b\sum_{i=1}^{k} |\,W(s_i)\,|\,\Delta_i[\Phi], \quad F_1 = N \\ (N = \max |\,\varphi(s)\,|, \;\; \Delta_i[\Phi] = |\,\Phi(s_i) - \Phi(s_i - 0)\,|) \end{gathered} \tag{2.3}$$

from which we have

$$F_{k+1} = F_k + b\,|\,W(s_k)\,|\,\Delta_k[\Phi] \leqslant F_k \exp\{b\Delta_k[\Phi]\}$$

and, consequently,

$$F_{k+1} \leqslant N \exp\{b \sum_{i=1}^{k} \Delta_i[\Phi]\} \tag{2.4}$$

Formulas (2.3) and (2.4) yield the estimate of solution of Eq.(2.1) on segment $[0, b]$

$$|\,W(s)\,| \leqslant N \exp\{bV[\Phi]\} \tag{2.5}$$

where $V[\Phi]$ is the total variation of function $\Phi(s)$ on segment $[0, b]$.

Let us now make only the assumption that $\Phi(s)$ is a function of limited variation continuous on the right. We approximate $\Phi(s)$ by the sequence of step functions $\Phi_n(s)$ $(n = 1, 2, \ldots)$ and construct respective solutions of $W_n(s)$ in the form

$$W_n(s) = \varphi(s) + \int_0^s (s - \alpha) W_n(\alpha)\, d\Phi_n(\alpha) \tag{2.6}$$

We select functions $\Phi_n(s)$ so that each of them as a finite number of discontinuities, that at discontinuity points values of $\Phi_n(s)$ and $\Phi(s)$ are the same, and that at every point of segment $[0, b]$ functions $\Phi_n(s)$ converge to $\Phi(s)$. Under these conditions the variations of functions $\Phi_n(s)$ and solutions of Eqs.(2.6) are uniformly bounded.

$$V[\Phi_n] \leqslant V[\Phi], \quad |W_n(s)| \leqslant N \exp\{bV[\Phi]\} \tag{2.7}$$

Moreover the sequence $W_n(s)$ $(n = 1, 2, \ldots)$ is equicontinuous. Indeed, from (2.6) we have

$$W_n(s_2) - W_n(s_1) = \varphi(s_2) - \varphi(s_1) + (s_2 - s_1)\int_0^{s_1} W_n(\alpha)\, d\Phi_n(\alpha) + \int_{s_1}^{s_2} (s_2 - \alpha) W_n(\alpha)\, d\Phi_n(\alpha)$$

Taking in this equality absolute values and evaluating the integrals using the mean value theorem for the Stieltjes integrals, we obtain

$$|W_n(s_2) - W_n(s_1)| \leqslant |\varphi(s_2) - \varphi(s_1)| + |s_2 - s_1| \times V[\Phi] N \exp\{bV[\Phi]\} \tag{2.8}$$

Applying the Arzela principle of compactness, we conclude that there is an infinite sequence of values of n such that the solution of Eqs.(2.6) uniformly converges to the limit function $W(s)$. Passing in equality (2.6) to limit with $n \to \infty$, we find /3/ that function $W(s)$ satisfies Eq.(2.1) and inequality (2.5). The theorem is proved.

Remark. If the conditions of Theorem 1 are satisfied and function $\varphi(s)$ has bounded variation on segment $[0, b]$, the solution of Eq.(2.1) has a bounded variation on $[0, b]$. By virtue of uniformity estimate (2.8) we have

$$V[W] \leqslant V[\Phi] + bV[\Phi] N \exp\{bV[\Phi]\}$$

Theorem 2. Let the conditions of Theorem 1 be satisfied, function $\varphi(s)$ have a bounded variation on segment $[0, b]$, and $\Phi_n(s)$ be a sequence continuous on the right functions of bounded variation on $[0, b]$ uniformly converging to $\Phi(s)$. Then solutions of Eqs.(2.6) are also uniformly convergent to the solution of Eq.(2.1).

Proof. Composing the remainder of Eqs.(2.1) and (2.6) we obtain

$$W(s) - W_n(s) = \varphi_n(s) + \int_0^s (s - \alpha)[W(\alpha) - W_n(\alpha)]\, d\Phi_n(\alpha) \tag{2.9}$$

$$\varphi_n(s) = \int_0^s (s - \alpha) W(\alpha)\, d[\Phi(\alpha) - \Phi_n(\alpha)]$$

Considering (2.9) as the equation concerning functions $U_n(s) = W(s) - W_n(s)$ and using the estimate (2.5) we obtain

$$|U_n(s)| \leqslant \max |\varphi_n(s)| \exp\{bV[\Phi_n]\} \tag{2.10}$$

The estimate of $|\varphi_n(s)|$ is obtained by integration by parts and, then, the mean value theorem. As the result, we have

$$|\varphi_n(s)| \leqslant b\{|W(0)||\Phi(0) - \Phi_n(0)| + \max|\Phi(s) - \Phi_n(s)|[V[W] + \max|W(s)|]\} \tag{2.11}$$

Owing to the uniform convergence of functions $\Phi_n(s)$ to $\Phi(s)$ the quantities $V[\Phi_n]$ are uniformly bounded and $\max|\varphi_n(s)| \to 0$ as $n \to \infty$. It now follows from inequality (2.10) that the solutions of Eqs.(2.6) uniformly converge to the solution of Eq.(2.1). The theorem is proved.

3. The input equation (1.9) is a particular case of the considered above Eq.(2.1) with functions

$$\varphi(s) = \frac{s\, dW_1(0)}{ds}, \quad \Phi(s) = r^2 \int_0^s \rho_1^2(\alpha)\, d\alpha + gc^{-2}\rho_1(s) \tag{3.1}$$

satisfying the conditions of the above proved Theorems 1 and 2.

It has been thus established that the integral equation (1.9) has a unique continuous solution for which estimate (2.5) is valid. Moreover, solutions of equations of the form (2.6) uniformly converge to the solution of Eq.(1.9), when the density approximation $\rho_n(s)$ $(n \geqslant 2)$ uniformly converges to function $\rho_1(s)$.

Then, applying the results of /3/ we formulate the following statement. The derivative in the right-hand side when $0 \leqslant s \leqslant b$ of solution of Eq.(1.9) is defined by equality (1.8) (for the continuous on the right function $\rho_1(s)$). Function $dW_1(s)/ds$ is the bilateral derivative of $W_1(s)$ at points where function $\rho_1(s)$ is continuous or $W_1(s) = 0$.

If function $\rho_n(s)$ uniformly converges to $\rho_1(s)$ on segment $[0, b]$ as $n \to \infty$, then also the derivatives $dW_n(s)/ds$ defined by the equalities

$$\frac{dW_n(s)}{ds} = \frac{dW_1(0)}{ds} + \int_0^s W_n(s)\, d\Phi_n(s), \tag{3.2}$$

$$\Phi_n(s) = r^2 \int_0^s \rho_n^2(\alpha)\, d\alpha + gc^{-2}\rho_n(s)$$

and (2.6) uniformly converge to the derivative of solution of Eq.(1.9).

Indeed, by constructing the remainder of equalities (1.8) and (3.2), we obtain

$$\frac{dW_1(s)}{ds} - \frac{dW_n(s)}{ds} = \int_0^s [W_1(\alpha) - W_n(\alpha)]\, d\Phi_n(\alpha) - \int_0^s W_1(\alpha)\, d[\Phi_n(\alpha) - \Phi(\alpha)] \tag{3.3}$$

Integration by parts the second integral in the right-hand side of formula (3.3) and application of the mean value theorem yield

$$\left|\frac{dW_1(s)}{ds} - \frac{dW_n(s)}{ds}\right| \leqslant \max|W_1(s) - W_n(s)|\, V[\Phi_n] + \max|\Phi(s) - \Phi_n(s)|\, \{\max|W_1(s)| + V[W_1]\} \tag{3.4}$$

whose right-hand side approaches zero as $n \to \infty$.

In the variance equation

$$dW_1/ds - gc^{-2}\rho_1 W_1 = 0 \quad (s = b)$$

and in Eqs.(1.8) and (1.9) appear two parameters: r and c. The estimates (2.10) and (3.4) are not uniform in the range of possible variation of parameters r and c. They are uniform only for bounded above r and bounded below phase velocity c. Analysis of the asymptotic behavior of function $c = c(r)$ as $r \to \infty$ shows that in a stepwise stratified fluid $c = O(r^{-1/2})$ /4/, while in fluid with a continuous mean density profile $c = O(r^{-1})$.

Example 1. Let the fluid density in the unperturbed state be defined by the law $\rho_0(z) = \rho_1 \exp(-kz)$, where $k > 0$. The analytic solution of the variance equation for this model was obtained in /1/ using the "solid cover" approximation

$$\sigma_j^2 = gkr^2 [r^2 + k^2/4 + \pi^2 H^{-2} (j-1)^2]^{-1} \tag{3.5}$$

where σ_j is the j-th frequency mode for the wavenumber r. We divide the fluid layer $-H \leqslant z \leqslant 0$ in n layers of equal thickness $h_n = H/n$ in each of which density is assumed constant and equal that of the middle layer $\rho_0(z)$. In this case the variational equation for a multilayer fluid under condition of "solid cover" is of the form

$$R_1(\sigma^2) = 0 \tag{3.6}$$

where R_1 is defined by the recurrent formula /4/

$$R_{n+1} = 0,\quad R_n = 1,\quad R_m = R_{m+1}[\lambda b_n(1+\gamma_n) - \varepsilon_n] - \gamma_n \lambda^2 (b_n^2 - \operatorname{th}^2 r) R_{m+2} \tag{3.7}$$

$$\lambda = (gr \operatorname{th} r)^{-1}\sigma^2,\quad \gamma_n = \exp(-kh_n),\quad \varepsilon_n = 1 - \gamma_n,\quad b_n = \operatorname{th} r \operatorname{cth} rh_n$$

Using (3.7) we prove by inducation the validity of formula

$$R_m = [\lambda^2 \gamma_n (b_n^2 - \operatorname{th}^2 r)]^{(n-m)/2} \frac{\sin[(n+1-m)(\pi/2 - \theta)]}{\cos\theta} \tag{3.8}$$

$$(m = n+1, n, \ldots, 1)$$

$$\theta = \arcsin\left[\frac{\lambda b_n (1+\gamma_n) - \varepsilon_n}{2\lambda\sqrt{\gamma_n (b_n^2 - \operatorname{th}^2 r)}}\right]$$

The zeros of $R_1(\theta)$ are obtained analytically. We have

$$\theta_j = \frac{\pi}{2} - \frac{\pi}{n}(j-1) \quad (j = 2, 3, \ldots, n)$$

Having determined θ_j we obtain exact variational equations for the multilayer model that approximate continuous stratification by the exponential one

$$\sigma_{nj}^2 = gr \operatorname{sh} rh_n \operatorname{sh} \frac{kn_n}{2} \left[\operatorname{ch} rh_n \operatorname{ch} \frac{kh_n}{2} - \cos \frac{\pi}{n}(j-1)\right]^{-1} \quad (j = 2, 3, \ldots, n) \tag{3.9}$$

whose expansion in series in parameter n^{-1} of function σ_{nj}^2, yields

$$\sigma_{nj}^2 = gkr^2 \left\{ \frac{1}{r^2 + \frac{k^2}{4} + \pi^2 H^{-2}(j-1)^2} - \frac{H^2}{12n^2}\left[1 - \frac{r^2k^2}{r^2 + \frac{k^2}{4} + \pi^2 H^{-2}(j-1)^2}\right] + O(n^{-4}) \right\} \tag{3.10}$$

The first term of this expansion is the same as in (3.5), hence

$$\lim_{n\to\infty} \sigma_{nj}^2 = \sigma_j^2$$

The second term of expansion (3.10) approaches zero uniformly with respect to r and j as $n\to\infty$, while the asymptotic behavior of σ_{nj}^2 and σ_j^2 differs as $r\to\infty$.

It follows from (3.5) and (3.9) that

$$\sigma_j^2 \approx gk, \quad \sigma_{nj}^2 \approx gr \operatorname{th} \frac{kh_n}{2} \quad (r \gg 1)$$

For the multilayer model the vertical velocity component of the j-th mode at points $z_m = -(m/n)H$ is determined by formula

$$W_{nj}(z_m) = \frac{\operatorname{sh} rh_n}{r} \operatorname{cosec}\left[\pi \frac{(j-1)}{n}\right] W_z(-H) \exp\left[kH \frac{(n-m)}{2n}\right] \times \sin\left[\pi(j-1)\frac{(n-m)}{n}\right]$$

and in the case of continous stratification

$$W_j(z_m) = \frac{H}{\pi(j-1)} W_z(-H) \exp\left[kH \frac{(n-m)}{2n}\right] \sin\left[\pi(j-1)\frac{(n-m)}{n}\right]$$

Obviously

$$\lim_{n\to\infty} W_{nj}(z_m) = W_j(z_m)$$

and this passing to limit is not uniform with respect to r and j.

Example 2. In /4/ for continuous density distribution on discontinuity layer formulas

$$\rho_0(z) = \begin{cases} \rho_1 & (-H_1 \leqslant z \leqslant 0) \\ \rho_1 \exp[-k(z+H_1)] & (-H_2 \leqslant z \leqslant -H_1) \\ \rho_1 \exp[k(H_2 - H_1)] & (-H_3 \leqslant z \leqslant -H_2) \end{cases} \tag{3.11}$$

$$H_1 = 20, \quad H_2 = 70, \quad H_3 = 2070 \text{ m}, \; k(H_2 - H_1) = 0.002$$

are given, and for stepwise distributions $\rho_n(z)$ that approximate $\rho_0(z)$ results of calculations of phase velocity propagation are given for long internal waves. Numerical calculations had shown /4/ that as $\rho_n(z)$ and $\rho_0(z)$ approach, the phase velocities in the case of multilayer model approach the phase velocities of the continuous mode. The convergence rate decreases with increasing mode number.

Table 1

r	$n = 12$	52	102	202	c
0.0	5.0	3.54	3.51	3.494	3.492
0.5	3.1	2.61	2.57	2.556	2.552
1.0	2.2	1.78	1.72	1.699	1.693
1.5	1.8	1.35	1.26	1.230	1.222
2.0	1.6	1.10	0.99	0.957	0.946

The variational dependence of phase velocity of the tenth internal mode on the wave number r are shown above for the same density distribution (the respective column is marked by letter c) and series meanings n of several multilayer approximations $\rho_n(z)$. Multilayer approximations are constructed as follows. The layer of discontinuity $(-H_2 \leqslant z \leqslant -H_1)$ was divided in $(n-2)$ layers of equal thickness, density $\rho_n(z)$ in each layer was assumed equal to the density $\rho_0(z)$ in the middle layer. The tabulated values in Table 1 are dimensinal: that of the wave number is m^{-1} and of phase velocities $cm.sec^{-1}$. The multilayer approximation yields in this example somewhat higher phase velocities which diminish monotonously as the number of layers is increased. The convergence rate decrease with increasing wavenumber r. However the multilayer approximation $\rho_0(z)$ enables the determination of phase velocity of the first ten internal modes with an accuracy of 5% in a fairly large range of wave lengths between 3m an infinity with $n = 102$.

We would also point out that the use of multilayer approximations of density distribution provides means for obtaining wave field characteristics in a form convenient for numerical computations in the form of recurrent formulas of the type (3.7).

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